Dec 23, 2016, 18:30 (Duration: 110 min)

MATH 101/1001 Calculus I Midterm-2

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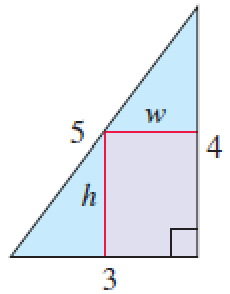
Department: __________________________________ **Student Number:** ___________

In solving the following problems, you are required to show all your work and provide the necessary explanations everywhere to get full credit.

Problem 1: [20 pts] Determine the dimensions (h and w) of the <u>rectangle of largest area</u> that can be inscribed in the right triangle shown in the figure. How do you know that the h and w values you find give the largest area and not the smallest area?



SOLUTION:

Let $(x, y) = \left(x, \frac{4}{3}x\right)$ be the coordinates of the corner that intersects the line. Then base $= 3 - x$ and height $= y = \frac{4}{3}x$, thus the area of the rectangle is given by $A = (3-x)\left(\frac{4}{3}x\right) = 4x - \frac{4}{3}x^2$, $0 \le x \le 3$. $A' = 4 - \frac{8}{3}x$, $A' = 0$ $\Rightarrow x = \frac{3}{2}$. $A'' = -\frac{4}{3} \Rightarrow A''\left(\frac{3}{2}\right) < 0 \Rightarrow$ local maximum at the critical point. The base $= 3 - \frac{3}{2} = \frac{3}{2}$ and the height $= \frac{4}{3}\left(\frac{3}{2}\right) = 2$.

Problem 2: [10 pts] Find the linearization *L(x)* at ***x = -1*** of

$$g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$$

SOLUTION:

$g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt \Rightarrow g'(x) = (\sec(x^2-1))(2x) = 2x\sec(x^2-1) \Rightarrow g'(-1) = 2(-1)\sec((-1)^2-1) = -2;$

$g(-1) = 3 + \int_1^{(-1)^2} \sec(t-1)\,dt = 3 + \int_1^1 \sec(t-1)\,dt = 3 + 0 = 3;$

$L(x) = -2(x-(-1)) + g(-1) = -2(x+1) + 3 = -2x + 1$

Problem 3: [20 pts] Evaluate the following integrals:

a) $\int_0^\pi \frac{sinx}{\sqrt{2+cosx}}dx$ b) $\int e^{3x} sin2xdx$

b) $\int_0^\pi \frac{sinx}{\sqrt{2+cosx}}dx$

Solution: a) Letting $2 + cosx = u$, $du = -sinxdx$. Then integral becomes $\int -\frac{du}{\sqrt{u}} = -2u^{1/2} = \left(-2\sqrt{2+cosx}\right)(0,\pi) = 2\sqrt{3} - 2$

b) $\int e^{3x} sin2xdx$

Solution: Let $u = sin2x$, $dv = e^{3x}dx$. Then $du = 2cos2xdx$, $v = \frac{e^{3x}}{3}$.

By integration by parts formula, given integral I will be equal to

$I = \frac{e^{3x}}{3} sin2x - \int \frac{e^{3x}}{3}\ 2cos2xdx = \frac{e^{3x}}{3} sin2x - \left[\frac{2}{3}\int e^{3x} cos2xdx\right]$.

By integration by parts again,

$$I = \frac{e^{3x}}{3} sin2x + \frac{2}{9} e^{3x} cos2x - \frac{4}{9}\int e^{3x} sin2xdx$$

Then, $I = \frac{9}{13}\left(\frac{e^{3x}}{3} sin2x + \frac{2}{9} e^{3x} cos2x\right) + c$

Problem 4: [20 pts] Let $f(x) = x^4 - 5x^2 + 6$.

a) Find the intervals on which f is increasing or decreasing. Find all local extrema for f.

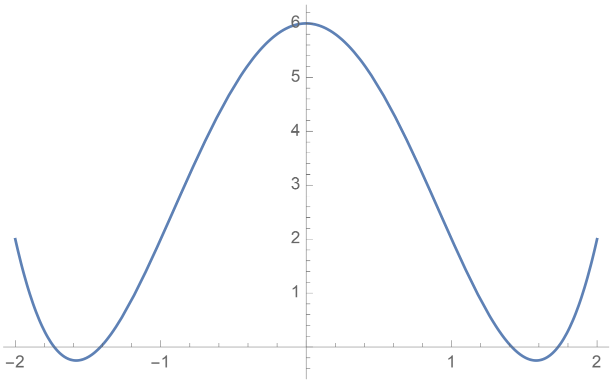
$f'(x) = 2x(-5 + 2x^2)$. f is increasing on $[-\sqrt{\frac{5}{2}}, 0]$ and $[\sqrt{\frac{5}{2}}, \infty)$, f is decreasing on $[-\infty, -\sqrt{\frac{5}{2}})$ and $[0, \sqrt{\frac{5}{2}}]$.

$f\left(\sqrt{\frac{5}{2}}\right) = f\left(-\sqrt{\frac{5}{2}}\right) = -\frac{1}{4}$ is a local minimum. $f(0) = 6$ is a local maximum.

b) Find the intervals on which f is concave up or down. Find all inflection points for f.

$f'(x) = 2(-5 + 6x^2)$. f is concave up on $[-\infty, \sqrt{\frac{5}{6}}]$ and $[\sqrt{\frac{5}{6}}, \infty)$, f is concave up on $[-\sqrt{\frac{5}{6}}, \sqrt{\frac{5}{6}}]$. Inflection points are $x = -\sqrt{\frac{5}{6}}$ and $x = \sqrt{\frac{5}{6}}$.

c) Sketch the graph of f using parts (a) and (b).



Problem 5: [10 pts] Let $y = (1/x)^{\ln x}$. Find $\frac{dy}{dx}$.

Using logarithmic differentiation, $\ln y = \ln x \ln(1/x) = -(\ln x)^2$. Hence $\frac{dy}{dx} = y\,(-2\ln x\,\frac{1}{x}) = -2(1/x)^{\ln x + 1}\ln x$.

Problem 6: [20 pts] Find the volume of the solid generated by revolving the region bounded by the curves $x^2 = 4y$ and $y = \frac{1}{2}x$ about the y-axis.

Solution:

The curves $x^2 = 4y$ and $y = \frac{1}{2}x$ intersect at (0,0) and (2,1).

Using the circular ring formula and integrating along the y-axis:

$$V = \int_0^1 \pi\left[\left(\sqrt{4y}\right)^2 - (2y)^2\right]dy = \pi\int_0^1 (4y - 4y^2)dy = \pi\left(2y^2 - \frac{4}{3}y^3\right)\Big|_0^1 = \frac{2\pi}{3}$$